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On Racah coefficients of the quantum algebra $U_q(su_2)$

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Abstract. A new expression for the Racah coefficients of the quantum algebra $U_q(su_2)$ is derived. New formulae for these coefficients can be obtained with the help of the results of the theory of basic hypergeometric functions. It is shown that the Clebsch-Gordan coefficients are recovered as the asymptotic limit of the Racah coefficients. Recurrence relations for the Racah and Clebsch-Gordan coefficients are derived.

1. Introduction

Quantum groups and algebras appeared in the quantum method of the inverse scattering problem (Drinfel'd 1986, Jimbo 1986, Kulish and Reshetikhin 1983). They are of great importance for applications in classical and quantum integral systems, in quantum field theory, in statistical physics, and in the theory of basic hypergeometric functions. The quantum algebra U_q (su_2) is intensively used in conformal field theory (Alvares-Gaume *et al* 1989, Moore and Seiberg 1989). Recently Biedenharn (1989a) and Macfarlane (1989) have considered a q-analogue of the quantum harmonic oscillator which is related to the algebra $U_q(su_2)$.

In order to apply the quantum algebra $U_q(su_2)$ in physics we need a well developed theory of its representations. It is important to have the good theory of the Clebsch-Gordan (CG) and Racah coefficients. These coefficients are related to tensor products of representations. Tensor products are used for construction of the universal R-matrices which are of great importance for Yang-Baxter equations (Jimbo 1985).

Kirillov and Reshetikhin (1988) have initiated study into the CG and Racah coefficients of $U_q(su_2)$. They gave three expressions for CG coefficients and one expression for Racah coefficients. The CG coefficients of $U_q(su_2)$ are also considered by Groza *et al* (1990), Koelink and Koornwinder (1990) and Vaksman (1989). Biedenharn (1989b) began studying the q-tensor operators which are related to the CG and Racah coefficients of $U_q(su_2)$.

The present paper is devoted to the Racah coefficients of the quantum algebra $U_q(su_2)$. In section 2 we describe this quantum algebra. Necessary information regarding the CG coefficients of $U_q(su_2)$ is given. In section 3 a new expression for the Racah coefficients of $U_q(su_2)$ is derived. It is shown how to obtain other expressions for these coefficients with the help of the results of the theory of basic hypergeometric functions. It turns out that the Racah coefficients of $U_q(su_2)$ are related in a simple way to the Racah coefficients of the classical algebra su_2.

New recurrence relations for the Racah coefficients are derived in section 4. In section 5 we show that the CG coefficients of $U_q(su_2)$ are obtained as the asymptotic limit of the Racah coefficients. This fact allows one to obtain the results for the CG

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coefficients from those for the Racah coefficients. Using this method we derive in section 6 a new expression for the CG coefficients.

We shall often refer to results on the classical CG and Racah coefficients (Beidenharn and Louck 1981, Edmonds 1957, Varshalovich *et al* 1975).

2. The quantum algebra $U_q(su_2)$

The quantum algebra $U_q(su_2)$ is an associative algebra generated by the elements J_+, J_-, $J_z \equiv J$ obeying the commutator relations

$$[J, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = \frac{q^J - q^{-J}}{q^{1/2} - q^{-1/2}}. \tag{1}$$

It is a deformation of the universal enveloping algebra of the Lie algebra su_2. In the limit $q \to 1$, the relations (1) transform to the well known formulae for the generators of su_2. The structure of a Hopf algebra is introduced into $U_q(su_2)$ (Drinfel'd 1986). This structure includes the homomorphism

$$\Delta: \qquad U_q(su_2) \to U_q(su_2) \otimes U_q(su_2)$$

which acts onto J_+, J_-, J as

$$\Delta(J_\pm) = J_\pm \otimes q^{J/2} + q^{-J/2} \otimes J_\mp \qquad \Delta(J) = J \otimes 1 + 1 \otimes J.$$

It means that the tensor product of representations of $U_q(su_2)$ is not commutative, $T_1 \otimes T_2 \neq T_2 \otimes T_1$.

As in the classical case, finite-dimensional representations T_l of $U_q(su_2)$ are given by integral or half-integral numbers l (Jimbo 1985). For the orthonormal basis $|l, m\rangle$, $m = -l, -l+1, \ldots, l$, we have

$$J_\pm |l, m\rangle = ([l \mp m][l \pm m + 1])^{1/2} |l, m \pm 1\rangle \qquad J|l, m\rangle = m|l, m\rangle$$

where $[n]$ means the term

$$[n] = [n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = [n]_{q^{-1}}. \tag{2}$$

For the tensor product $T_{l_1} \otimes T_{l_2}$ we have (Jimbo 1985)

$$T_{l_1} \otimes T_{l_2} = \sum_l \oplus T_l$$

where the summation is over $l = |l_1 - l_2|, |l_1 - l_2| + 1, \ldots, l_1 + l_2$. As in the classical case, the CG coefficients of this tensor product are defined by the relation

$$|l_1, j\rangle |l_2, k\rangle = \sum_{l,m} C^{l_1 l_2 l}_{jkm} |l, m\rangle.$$

If $j + k \neq m$ then $C^{l_1 l_2 l}_{jkm} = 0$. The CG coefficients constitute the unitary matrix. Therefore, the orthogonality relations are valid, which have the same form as in the classical case.

Below we shall use the expression (Vaksman 1989)

$$C^{abc}_{jkm} = q^A \left(\frac{[a-j]![b-k]![c-m]![c+m]![a+b-c]![2c+1]}{[a+j]![b+k]![a-b+c]![a+b+c+1]![c-a+b]!} \right)^{1/2} \times \sum_r \frac{(-1)^{r+a-j} q^{r(c+m+1)/2} [a+j+r]![b+c-j-r]!}{[r]![c-m-r]![a-j-r]![b-c+j+r]!} \tag{3}$$

where for brevity the symbols a, b, c are used instead of l_1, l_2, l. Here the q-factorials $[n]!$ are defined as

$$[n]! = [1][2]\dots[n]$$

where $[k]$ is given by (2), and

$$A = -\tfrac{1}{4}a(a+1) + \tfrac{1}{4}b(b+1) - \tfrac{1}{4}c(c+1) + \tfrac{1}{2}j(m+1).$$

The summation in (3) is over those values of r for which the numbers n appearing in the factorials $[n]!$ are non-negative. It is easy to find that

$$C^{abc}_{jkc} = \frac{(-1)^{a-j} q^B \Delta(a,b,c)}{[a-b+c]![b-a+c]!} \left(\frac{[a+j]![b+k]![2c+1]}{[a-j]![b-k]!} \right)^{1/2} \tag{4}$$

where $j+k=c$, $B = \frac{1}{4}\{b(b+1) - a(a+1) - c(c+1) + 2j(c+1)\}$ and

$$\Delta(a,b,c) = \left(\frac{[a+b-c]![a-b+c]![b-a+c]!}{[a+b+c+1]!} \right)^{1/2}.$$

For the tensor product $T_l \otimes T_{1/2}$ we have

$$C^{l,1/2,l+1/2}_{j,\pm 1/2, j\pm 1/2} = q^{\pm(l\mp j)/4} \left(\frac{[l \pm j + 1]}{[2l+1]} \right)^{1/2} \tag{5}$$

$$C^{l,1/2,l-1/2}_{j,\pm 1/2, j\pm 1/2} = \mp q^{\mp(l\pm j+1)/4} \left(\frac{[l \mp j]}{[2l+1]} \right)^{1/2}. \tag{6}$$

The CG coefficients of $U_q(su_2)$ can be expressed in terms of finite basic hypergeometric series ${}_3\Phi_2$ (Groza *et al* 1990). Basic hypergeometric functions are defined as

$${}_{n+1}\Phi_n(a_1, a_2, \dots, a_{n+1}; b_1, b_2, \dots, b_n; q, z) = \sum_r \frac{(q^{a_1}; q)_r \dots (q^{a_{n+1}}; q)_r}{(q^{b_1}; q)_r \dots (q^{b_n}; q)_r} \frac{z^r}{(q;q)_r} \tag{7}$$

where

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j) \qquad (a;q)_0 = 1.$$

The detailed presentation of the theory of basic hypergeometric functions is given by Exton (1983) and by Gasper and Rahman (1989). Let us note that $(q^a; q)_s$ is related to $[n]!$ by the equations

$$(q^{N+1}; q)_s = \frac{[N+s]!}{[N]!} q^{s(s+2N-1)/4} (1-q)^s$$

$$(q^{-N}; q)_s = \frac{[N]!}{[N-s]!} (-1)^s q^{s(s-2N-3)/4} (1-q)^s.$$

3. The Racah coefficients of $U_q(su_2)$

The Racah coefficients appear in a consideration of the tensor product $T_{l_1} \otimes T_{l_2} \otimes T_{l_3}$. The product is associative, i.e.

$$(T_{l_1} \otimes T_{l_2}) \otimes T_{l_3} = T_{l_1} \otimes (T_{l_2} \otimes T_{l_3}).$$

As in the classical case, it leads to two orthonormal bases of the carrier space of $T_{l_1}\otimes T_{l_2}\otimes T_{l_3}$. The first basis consists of the vectors

$$|(l_1l_2)l_{12}, l_3; l, p\rangle = \sum_{i,j,k} C^{l_1l_2l_{12}}_{ijm} C^{l_{12}l_3l}_{mkp}(|l_1, i\rangle|l_2, j\rangle)|l_3, k\rangle$$

and the second one of the vectors

$$|l_1(l_2l_3)l_{23}; l, p\rangle = \sum_{i,j,k} C^{l_2l_3l_{23}}_{jkn} C^{l_1l_{23}l}_{inp}|l_1, i\rangle(|l_2, j\rangle|l_3, k\rangle)$$

where $i+j=m$, $m+k=p$, $j+k=n$ and $i+n=p$. These vectors are related by the unitary matrix R:

$$|(l_1l_2)l_{12}, l_3; l, p\rangle = \sum_{l_{23}} R(l_1l_2l_3, l_{12}l_{23}, l)|l_1(l_2l_3)l_{23}; l, p\rangle.$$

The numbers $R(l_1l_2l_3, l_{12}l_{23}, l)$ are called the Racah coefficients of the algebra $U_q(su_2)$. These coefficients obey the orthogonality relations which have the same form as in the classical case.

The Racah coefficients are connected with the $6j$-symbols

$$\begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix} = (-1)^{l_1+l_2+l_3+l}([2l_{12}+1][2l_{23}+1])^{-1/2} R(l_1l_2l_3, l_{12}l_{23}, l).$$

By the same reasoning as in the classical case, we obtain the relations

$$\delta_{ee'}R(abd, cf, e) = \sum_{i=-a}^{a} \sum_{j=-b}^{b} \sum_{k=-c}^{c} C^{abc}_{ijm} C^{cde'}_{mkp} C^{bdf}_{jkn} C^{afe}_{inp} \tag{8}$$

where $m=i+j$, $p=m+k$, $n=j+k$, $p=i+n$, and

$$\sum_{i=-a}^{a} \sum_{j=-b}^{b} C^{abc}_{i,j,m} C^{bdf}_{j,k,j+k} C^{afe}_{i,j+k,m+k} = R(abd, cf, e) C^{cde}_{m,k,m+k} \tag{9}$$

where $i+j=m$. The formula (8) shows that the Racah coefficients are real if the CG coefficients are real.

The formula (9) is used for derivation of expressions for the Racah coefficients. By using formula (9) and the Racah method (cf Biedenharn and Louck 1981), Kirillov and Reshetikhin (1988) have derived a q-analogue of the Racah formula for the Racah coefficients. This method is long and complicated. We derive other expressions for the Racah coefficients. Our method is simpler.

Let us put $i+j=c$ and $e-c=k$ into (9). We obtain

$$R(abd, cf, e) = (C^{cde}_{c,e-c,e})^{-1} \sum_{i+j=c} C^{abc}_{ijc} C^{bdf}_{j,e-c,j+e-c} C^{afe}_{i,j+e-c,e}.$$

Substituting the expressions for $C^{cde}_{c,e-c,e}$, C^{abc}_{ijc} and $C^{afe}_{i,j+e-c,e}$ from formula (4) and using the $6j$-symbols, we have

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{b-a+d+e}\left(\frac{[a+b-c]![a+f-e]![c+d+e+1]!}{[a+b+c+1]![a+c-b]![b+c-a]!}\right.$$
$$\times \left.\frac{[c+e-d]![2f+1]^{-1}}{[a+f+e+1]![a+e-f]![e+f-a]!}\right)^{1/2} q^D \sum_i q^{i(c+e+2)/2}$$
$$\times \left(\frac{[b+c-i]![e+f-i]!}{[b-c+i]![f-e+i]!}\right)^{1/2} \frac{[a+i]!}{[a-i]!} C^{bdf}_{c-i,e-c,e-i}$$

where

$$D=\tfrac{1}{4}\{b(b+1)+f(f+1)-d(d+1)-2a(a+1)-2c(e+1)\}.$$

Substituting here the expression (3) for the CG coefficients, we have

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}=\frac{(-1)^{2b-a+d+e-c}\Delta(a,b,c)\Delta(a,f,e)\Delta(b,d,f)}{[a+c-b]![b+c-a]![a+e-f]![f+e-a]![d+e-c]!}$$
$$\times\frac{\Delta(c,d,e)[c+d+e+1]!}{[b+f-d]![d+f-b]!}\sum_{i,r}\frac{(-1)^{i+r}q^F[a+i]![b+c-i+r]!}{[r]![a-i]![b-c+i-r]!}$$
$$\times\frac{[d-c+f+i-r]![f+e-i]!}{[d+c-f-i+r]![f-e+i-r]!}$$

where $F=-\frac{1}{2}a(a+1)+\frac{1}{2}r(e+f-i+1)+\frac{1}{2}i(i+1)$. Now we replace the summations over r and i by summations over $k=i-r$ and i. The sum over i is reduced to

$${}_2\Phi_1(-a+k,a+k+1;-f-e+k;q,q).$$

This basic hypergeometric series can be summed according to the formula (3.3.2.7) from Slater (1966). Therefore we obtain

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}=\frac{(-1)^{b-a+d+e}\Delta(a,b,c)\Delta(a,f,e)\Delta(b,d,f)}{[a-b+c]![b-a+c]![a-f+e]![b-d+f]![d-b+f]!}$$
$$\times\frac{\Delta(c,d,e)[a+e+f+1]![c+d+e+1]!}{[d+e-c]!}\sum_s\frac{(-1)^s[2b-s]!}{[a+b-c-s]![s]!}$$
$$\times\frac{[a-b+c+s]![d+f-b+s]!}{[b+d-f-s]![c+f-b-e+s]![c+f-b+e+s+1]!}\tag{10}$$

where $s=b-c+k$.

By using formula (7) the sum in (10) can be expressed in terms of the finite basic hypergeometric function ${}_4\Phi_3$. We have

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}=\frac{(-1)^{b+d+e-a}\Delta(a,b,c)\Delta(a,f,e)\Delta(b,d,f)}{[a+b-c]![b+c-a]![a+e-f]![d+e-c]![b+d-f]!}$$
$$\times\frac{\Delta(c,d,e)[2b]![c+d+e+1]![a+e+f+1]!}{[b+f-d]![c+f-b-e]![c+f+e-b+1]!}$$
$$\times{}_4\Phi_3\left(\begin{matrix}a-b+c+1,c-a-b,d-b+f+1,f-b-d\\-2b,c+f-b-e+1,c+f+e-b+2\end{matrix}\middle| q,q\right).\tag{11}$$

The other expressions for the $6j$-symbols are obtained from this one with the help of the relation

$${}_4\Phi_3(-n,a,b,c;d,e,f;q,q)$$
$$=q^{(b+c-d)n}\frac{(q^{a-e-n+1};q)_n(q^{a-f-n+1};q)_n}{(q^e;q)_n(q^f;q)_n}$$
$$\times{}_4\Phi_3(-n,a,d-b,d-c;d,a-e-n+1,a-f-n+1;q,q)$$

which is valid for $a+b+c-n+1=d+e+f$ (cf formula (1.28) of Askey and Wilson 1985). By using this relation we have obtained q-analogues of all known expressions for the classical $6j$-symbols. All these q-analogues can be obtained from the classical formulae by replacing all factorials $m!$ by the q-factorials $[m]!$ (if the expressions are given in the form of a sum) and, in addition, replacing ${}_4F_3(a, b, c, d; e, f, g; 1)$ by ${}_4\Phi_3(a, b, c, d; e, f, g; q, q)$ (if the $6j$-symbols are expressed by means of a hypergeometric function).

According to formula (2)

$$[n]_q = [n]_{q^{-1}}.$$

Therefore, the $6j$-symbols coincide for the algebras $U_q(su_2)$ and $U_{q^{-1}}(su_2)$:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_q = \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_{q^{-1}}.$$

The expressions for the classical $6j$-symbols transform into each other under symmetry relations. Since the $6j$-symbols for $U_q(su_2)$ are obtained from those for su_2 by the changing of factorials, then exactly the same transformations are valid for the q-analogue $6j$-symbols. Hence, the symmetry relations for the $6j$-symbols of $U_q(su_2)$ coincide with classical relations. A part of these symmetries is given by Kirillov and Reshetikhin (1988). Let us write down the 'reflection' symmetries, which are absent in Kirillov and Reshetikhin (1988):

$$\begin{aligned}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} &= -\begin{Bmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{d} & \bar{e} & \bar{f} \end{Bmatrix} \\ &= (-1)^n \begin{Bmatrix} \bar{a} & b & c \\ d & e & f \end{Bmatrix} \\ &= (-1)^m \begin{Bmatrix} \bar{a} & b & c \\ \bar{d} & e & f \end{Bmatrix} \\ &= (-1)^r \begin{Bmatrix} \bar{a} & \bar{b} & c \\ \bar{d} & e & f \end{Bmatrix}.\end{aligned}$$

Here

$$n = b - c - e + f \qquad m = 2(a + d) \qquad r = 2(c + f) + 1$$

and $\bar{g}$ means that in an expression for the $6j$-symbol g is replaced by $\bar{g} = -g - 1$. Expressions of the form $[-p]!/[-g]!$ with positive integers p and g appear under this replacement. They have to be replaced by $(-1)^{p-g}[g-1]!/[p-1]!$.

Below we shall use explicit expressions for the $6j$-symbols with one of the parameters equal to $\frac{1}{2}$. They have the form

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c+\frac{1}{2} & b+\frac{1}{2} \end{Bmatrix} = (-1)^{a+b+c+1} \left(\frac{[a+b+c+2][b-a+c+1]}{[2b+1][2c+1][2b+2][2c+2]} \right)^{1/2} \tag{12}$$

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c+\frac{1}{2} & b-\frac{1}{2} \end{Bmatrix} = (-1)^{a+b+c} \left(\frac{[a+b-c][a-b+c+1]}{[2b+1][2c+1][2b][2c+2]} \right)^{1/2} \tag{13}$$

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c-\frac{1}{2} & b+\frac{1}{2} \end{Bmatrix} = (-1)^{a+b+c} \left(\frac{[a+b-c+1][a-b+c]}{[2b+1][2c+1][2b+2][2c]} \right)^{1/2} \tag{14}$$

$$\begin{Bmatrix} a & b & c \\ \frac{1}{2} & c-\frac{1}{2} & b-\frac{1}{2} \end{Bmatrix} = (-1)^{a+b+c} \left(\frac{[a+b+c+1][b-a+c]}{[2b+1][2c+1][2b][2c]} \right)^{1/2}. \tag{15}$$

4. Recurrence relations for the Racah coefficients

The formula

$$R_n(\mu(x); a, b, c, d|q)$$
$$= {}_4\Phi_3(-n, a+b+n+1, -x, c+d+x+1; a+1, b+d+1, d+1; q, q)$$

defines a polynomial in $\mu(x) = q^{-x} + q^{c+d+x+1}$, which is called a q-Racah polynomial (cf Askey and Wilson 1979). Our notation is somewhat different from that of Askey and Wilson (1979).

Using in (11) the symmetry which transforms

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \quad \text{into} \quad \begin{Bmatrix} a & b & \bar{c} \\ d & e & \bar{f} \end{Bmatrix}$$

and introducing the notation

$$n = b - d + f \qquad x = b + c - a \qquad a = -2b - 1 \qquad b = -2f - 1$$
$$c = -b - c - e - f - 2 \qquad d = -b - c + e + f$$

we express the $6j$-symbol in terms of a q-Racah polynomial:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{b+c+f+e} \frac{\Delta(a, b, c)\Delta(a, e, f)\Delta(c, e, d)}{[a+b-c]![b+c-a]![d+c-e]![c+e-d]!}$$
$$\times \frac{\Delta(d, b, f)[2b]![b+f+c+e+1]![b+f+c-e]!}{[a+f-e]![f+e-a]![b+d-f]![b+f-d]!}$$
$$\times R_{b+f-d}(q^{a-b-c}(1-q^{-2a-1}); -2b-1, -2f-1,$$
$$-b-c-e-f-2, -b-c+f+e|q).$$

There is the recurrence relation (4.6) of Askey and Wilson (1979) for q-Racah polynomials. By transforming q-Racah polynomials into the $6j$-symbols we obtain the recurrence formula

$$A\begin{Bmatrix} a & b & c \\ d+1 & e & f \end{Bmatrix} - B\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} + C\begin{Bmatrix} a & b & c \\ d-1 & e & f \end{Bmatrix} = 0 \tag{16}$$

where

$$A = [2d]([c+d+e+2][c+e-d][d+e-c+1][d+c-e+1][b+f-d]$$
$$\times [b+d-f+1][b+d+f+2][d+f-b+1])^{1/2}$$
$$B = [2d+2][d+c-e][d+e+c+1][b+d-f][b+d+f+1]$$
$$+ [2d][d+e-c+1][c+e-d][b+f-d][d+f-b+1]$$
$$- [2d][2d+1][2d+2][a+b+c+1][b+c-a]$$
$$C = [2d+2]([d+e+c+1][d+c-e][c+e-d+1][e-c+d]$$
$$\times [b+d+f+1][b+f-d+1][b+d-f][d+f-b])^{1/2}.$$

In order to obtain recurrence relations in which the parameters are changed by $\frac{1}{2}$, we use the q-analogue of the Biedenharn–Elliott identity (Kirillov and Reshetikhin 1988). It can be written as

$$R(a'ab, c'e, b')R(a'ed, b'c, d') = \sum_f R(abd, ef, c)R(c'bd, b'f, d')R(a'af, c'c, d').$$

In terms of the $6j$-symbols it takes the form

$$\sum_f (-1)^p[2f+1]\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}\begin{Bmatrix} c' & b & b' \\ d & d' & f \end{Bmatrix}\begin{Bmatrix} c' & a & a' \\ c & d' & f \end{Bmatrix}$$

$$=\begin{Bmatrix} c' & a & a' \\ e & b' & b \end{Bmatrix}\begin{Bmatrix} a' & e & b' \\ d & d' & c \end{Bmatrix} \tag{17}$$

where $p=a'+b'-c'-d'-a-b-c-d+e-f$. Put $c'=\frac{1}{2}$ into (17). Then $a'=a\pm\frac{1}{2}$, $b'=b\pm\frac{1}{2}$, $f=d'\pm\frac{1}{2}$. If $a'=a-\frac{1}{2}$, $b'=b-\frac{1}{2}$, then with the help of formulae (12), (14) and (15) we obtain the three-term recurrence relation

$$([a+b+c+1][b-a+c][d+e+c+1][d+c-e])^{1/2}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$

$$=([a+b-c+1][a-b+c][e+d-c+1][e+c-d])^{1/2}\begin{Bmatrix} a & b & c-1 \\ d & e & f \end{Bmatrix}$$

$$-[2c]([b+d+f+1][b+d-f])^{1/2}\begin{Bmatrix} a & b-\frac{1}{2} & c-\frac{1}{2} \\ d-\frac{1}{2} & e & f \end{Bmatrix}. \tag{18}$$

In the same way other three-term recurrence formulae can be obtained.

5. Clebsch-Gordan coefficients as the limit of Racah coefficients

Let us show that, as in the classical case, the CG coefficients of $U_q(su_2)$ are obtained as the asymptotic limit of the Racah coefficients. We shall follow the reasoning of Biedenharn and Louck (1981) for the classical case. Let us introduce the notation

$$R^{abc}_{jkm}(r)=R(r-m,a,b;r-k,c;r). \tag{19}$$

The inverse relation is

$$R(abd,ef,c)=R^{bdf}_{e-a,c-e,c-a}(c). \tag{20}$$

We consider $R^{abc}_{jkm}(r)$ for all values of j, k, m and a, b, c, for which the CG coefficients C^{abc}_{jkm} have a sense, and put $R^{abc}_{jkm}(r)=0$ for additional values of the parameters.

The formulae (12)-(15) can be written as

$$R^{a,1/2,a+1/2}_{j,\pm1/2,j\pm1/2}(r)=\left(\frac{[a\pm j+1][2r\pm a-j+1]}{[2a+1][2r+1]}\right)^{1/2} \tag{21}$$

$$R^{a,1/2,a-1/2}_{j,\pm1/2,j\pm1/2}(r)=\mp\left(\frac{[a\mp j][2r\mp a-j+1\mp1]}{[2a+1][2r+1]}\right)^{1/2}. \tag{22}$$

Since

$$\lim_{r\to\infty}\frac{[a+r]}{[b+r]}=q^{(a-b)/2} \qquad |q|>1 \tag{23}$$

then it follows from (5), (6), (21) and (22) that for $k,m=-\frac{1}{2},\frac{1}{2}$ we have

$$\lim_{r\to\infty}R^{a,1/2,a+m}_{j,k,j+k}(r)=C^{a,1/2,a+m}_{j,k,j+k}. \tag{24}$$

Let us show that

$$\lim_{r\to\infty} R^{abc}_{jkm}(r) = C^{abc}_{jkm} \tag{25}$$

for all b, where C^{abc}_{jkm} are the CG coefficients of $U_q(su_2)$.

With the help of the orthogonality relations for the Racah coefficients we derive from the Biedenharn–Elliott identity:

$$\delta_{fg}R(a'af, c'c, d') = \sum_{e,e'} R(abd, eg, c)R(c'bd, e'f, d')R(a'ab, c'e, e')R(a'ed, e'c, d').$$

Putting here $g=f$, $d=\frac{1}{2}$, $b=f-\frac{1}{2}$ and using the Racah coefficients (21) and (22), we obtain the recurrence relation

$$\begin{aligned} R^{abc}_{jkm}(r) = &\left(\frac{[a+b-c][a-b+c+1][b-k][2r-b-k+1][c+m+1][2r+c-m+2]}{[2b][2c+1][2b][2r+1][2c+2][2r+1]}\right)^{1/2} \\ &\times R^{a,b-1/2,c+1/2}_{j,k+1/2,m+1/2}(r+\tfrac{1}{2}) \\ &+\left(\frac{[b-a+c][a+b+c+1][b-k][2r-b-k+1][c-m][2r-c-m+1]}{[2b][2c+1][2b][2r+1][2c][2r+1]}\right)^{1/2} \\ &\times R^{a,b-1/2,c-1/2}_{j,k+1/2,m+1/2}(r+\tfrac{1}{2}) \\ &-\left(\frac{[a+b-c][a-b+c+1][b+k][2r+b-k+1][c-m+1][2r-c-m]}{[2b][2c+1][2b][2r+1][2c+2][2r+1]}\right)^{1/2} \\ &\times R^{a,b-1/2,c+1/2}_{j,k-1/2,m-1/2}(r-\tfrac{1}{2}) \\ &+\left(\frac{[b-a+c][a+b+c+1][b+k][2r+b-k+1][c-m+1][2r-c-m]}{[2b][2c+1][2b][2r+1][2c][2r+1]}\right)^{1/2} \\ &\times R^{a,b-1/2,c-1/2}_{j,k-1/2,m-1/2}(r-\tfrac{1}{2}) \end{aligned} \tag{26}$$

where $j+k=m$. This relation generates all Racah coefficients if we start from the coefficients (21) and (22). Hence, as in the classical case (Biedenharn and Louck 1981), the Biedenharn–Elliott identity, the orthogonality relations for the Racah coefficients (which were used for obtaining relation (25)), and the coefficients (21) and (22) uniquely define all the Racah coefficients of the quantum algebra $U_q(su_2)$.

Using the limit (23) in (26) we receive the recurrence relation

$$\begin{aligned} \hat{C}^{abc}_{jkm} = &\, q^{(t-s+1)/4}\left(\frac{[a+b-c][a-b+c+1][b-k][c+m+1]}{[2b][2c+1][2b][2c+2]}\right)^{1/2} \hat{C}^{a,b-1/2,c+1/2}_{j,k+1/2,m+1/2} \\ &+q^{-(p+s)/4}\left(\frac{[b-a+c][a+b+c+1][b-k][c-m]}{[2b][2c+1][2b][2c]}\right)^{1/2} \hat{C}^{a,b-1/2,c-1/2}_{j,k+1/2,m+1/2} \\ &-q^{(-t-s-1)/4}\left(\frac{[a+b-c][a-b+c+1][b+k][c-m+1]}{[2b][2c+1][2b][2c+2]}\right)^{1/2} \hat{C}^{a,b-1/2,c+1/2}_{j,k-1/2,m-1/2} \\ &+q^{(p-s)/4}\left(\frac{[b-a+c][a+b+c+1][b+k][c+m]}{[2b][2c+1][2b][2c]}\right)^{1/2} \hat{C}^{a,b-1/2,c-1/2}_{j,k-1/2,m-1/2} \end{aligned} \tag{27}$$

for

$$\hat{C}^{abc}_{jkm} \equiv \lim_{r\to\infty} R^{abc}_{jkm}(r)$$

where $s=j+2k$, $t=c-b$, $p=b+c$.

Let us show that the CG coefficients C^{abc}_{jkm} satisfy the relation (27). Using the orthogonality relations for the CG coefficients we write formula (8) in the form

$$R(abd, cf, e)C^{afe}_{j,k,j+k}=\sum_{i,p} C^{abc}_{j,i,j+i}C^{cde}_{j+i,p,j+k}C^{bdf}_{ipk}.$$

Multiplying both sides by $R(abd, cf', e)$, summing over c and using the orthogonality relation for the Racah coefficients, we get

$$\delta_{ff'}C^{afe}_{j,k,j+k}=\sum_{i,p,c} R(abd, cf, e)C^{abc}_{j,i,j+i}C^{cde}_{j+i,p,j+k}C^{bdf'}_{ipk}.$$

Now we set $f=f'$, $d=\frac{1}{2}$, $b=f-\frac{1}{2}$ and use formulae (5), (6), (21) and (22) for special CG and Racah coefficients. Replacing f by b, and e by c, we obtain the relation (27) for C^{abc}_{jkm}. Since by (24)

$$\hat{C}^{a,1/2,c}_{j,k,j+k}=C^{a,1/2,c}_{j,k,j+k}$$

and relation (27) defines all $\hat{C}^{abc}_{jkm}$ in terms of $\hat{C}^{a,1/2,c}_{jkm}$, then

$$\hat{C}^{abc}_{jkm}=\lim_{r\to\infty} R^{abc}_{jkm}(r)=C^{abc}_{jkm}$$

for all C^{abc}_{jkm}.

Thus, all the CG coefficients of the algebra $U_q(su_2)$ can be obtained as the asymptotic limit of the Racah coefficients of this algebra. As in the classical case (Biedenharn and Louck 1981), the Biedenharn–Elliott identity, the orthogonality relations for the Racah coefficients and the special Racah coefficients $R^{a,1/2,a+1/2}_{j,-1/2,j-1/2}$ completely define all the Racah coefficients and all the CG coefficients of the quantum algebra $U_q(su_2)$. But a derivation of explicit expressions for the coefficients by this method is not easy.

6. Clebsch–Gordan coefficients of $U_q(su_2)$

The limit (25) allows one to obtain expressions for the CG coefficients from those for the Racah coefficients. Using formula (23) we realise the limit transformation in the expression (10) for the Racah coefficients. As a result we obtain the expression for the CG coefficients of $U_q(su_2)$, which can be written as

$$C^{abc}_{jkm}=\frac{(-1)^{a-j}q^{H}\Delta(a,b,c)[2a]!}{[a-b+c]![b-a+m]![a+b-c]!}\left(\frac{[b-k]![b+k]![c+m]![2c+1]}{[a-j]![a+j]![c-m]!}\right)^{1/2}$$
$$\times\,{}_3\Phi_2(-a+j, b-a-c, b-a+c+1; -2a, b-a+m+1; q, q)$$

where $m=j+k$ and

$$H=-\tfrac{1}{4}(a-b+c)(b-a+c+1)+\tfrac{1}{2}j(b+1)+\tfrac{1}{2}ak.$$

Using the relation

$${}_3\Phi_2(-n, a, b; c, d; q, q)=\frac{(q^{c-a}; q)_n}{(q^c; q)_n}q^{na}{}_3\Phi_2(-n, a, d-b; a-c-n+1, d; q, q^{b-c+1})$$

we can obtain other expressions for the CG coefficients of $U_q(su_2)$ (cf Groza *et al* 1990).

If the limit (25) is fulfilled in formula (16) then the result is the recurrence relation

$$A_aC^{a+1,b,c}_{jkm}-B_aC^{abc}_{jkm}+C_aC^{a-1,b,c}_{jkm}=0,$$

where

$$A_a=[2a]([a-b+c+1][a+b-c+1][b+c-a][a+b+c+2][a-j+1][a+j+1])^{1/2}$$

$$\begin{aligned}B_a=q^{(c+k)/2}&[2a][2a+1][2a+2][c-m]\\&-q^{-(a+1)/2}[2a][a+b-c+1][b+c-a][a+j+1]\\&-q^{a/2}[2a+2][a+b+c+1][a+c-b][a-j]\end{aligned}$$

$$C_a=[2a+2]([a+b-c][a+c-b][b+c-a+1][a+b+c+1][a-j][a+j])^{1/2}.$$

Realising the same limit in (18), we obtain the three-term recurrence formula

$$\begin{aligned}([a+b+c+1][a+b-c])^{1/2}C^{a-1/2,b-1/2,c}_{j-1/2,k+1/2,m}&=q^{(a+b-j+k+1)/4}([a+j][b-k])^{1/2}C^{abc}_{jkm}-q^{-(a+b+j-k-1)/4}\\&\quad\times([a-j+1][b+k+1])^{1/2}C^{abc}_{j-1,k+1,m}.\end{aligned}$$

7. Conclusion

The CG and Racah coefficients of the algebra $U_q(su_2)$ describe the tensor product $T_a\otimes T_b$ and the recoupling in $T_a\otimes T_b\otimes T_c$, respectively. As in the classical case, the last tensor product is associative, i.e.

$$(T_a\otimes T_b)\otimes T_c=T_a\otimes(T_b\otimes T_c).$$

In accordance with this, the expressions for the Racah coefficients of $U_q(su_2)$ coincide up to the replacement of factorials $m!$ by q-factorials $[m]!$ with the expressions for the classical Racah coefficients. The product $T_a\otimes T_b$ is not commutative for $U_q(su_2)$. And the CG coefficients of $U_q(su_2)$ cannot be obtained from the CG coefficients of su_2 in the same way as in the case of the Racah coefficients. Nevertheless, as in the classical case, the CG coefficients of the algebra $U_q(su_2)$ are obtained from the Racah coefficients of this algebra by the limit procedure.

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